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BOUNDARY-LAYER MOMENTUM EQUATIONS FOR THREE-DIMENSIONAL FLOW

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SUMMARY

Boundary-layer momentum equations for the three-dimensional flow of a fluid with variable density and viscosity are presented in a form similar to the momentum equation for two-dimensional flow. The momentum equations can be reduced to the forms of the three-dimensional momentum equations that have been given recently by Prandtl for a fluid with constant density and viscosity. When the flow becomes two-dimensional, the momentum equation first given by von Kármán results. For flow in a convergent or divergent channel the equations reduce to the equations previously given by A. Kehl for a fluid with constant density and viscosity.

INTRODUCTION

Recently there has been an awakening of interest in the problem of three-dimensional boundary-layer flow; that is, flow where the velocity and static pressure outside the boundary layer are functions of two independent variables. In the usual two-dimensional boundary-layer theory for flow over slightly curved surfaces, or over bodies of revolution, the velocity and static pressure outside the boundary layer are functions of only one independent variable.

A case of three-dimensional boundary-layer flow that is of particular interest at present is the flow over sweptback wings for which the outer flow velocity and pressure gradient have a component in the direction of the chord and a component at right angles to the chord and in the direction of the span.

Except for a paper by Prandtl (reference 1) that recently became available, no literature concerned with the theoretical aspect of the problem is known. After giving a form of the boundary-layer momentum equation for the three-dimensional flow of an incompressible fluid with constant viscosity, Prandtl discusses a program based on the momentum equation. The program has as its goal the formulation of a method for computing the characteristics of laminar boundary layers in three dimensions that is similar to the Pohlhausen method for two-dimensional flow (reference 2) and a method for determining the characteristics of turbulent boundary layers that is based on experimental data.

The computing methods for both laminar and turbulent boundary layers in three-dimensions should be able to use a boundary-layer momentum equation in the same manner that approximate methods for computing laminar and turbulent boundary-layer characteristics in two-dimensional flow (reference 2, 3, and 4) use the von Kármán momentum equation (reference 5). The momentum equation, in addition to serving as a basis for approximate methods, should also suggest parameters to be constructed from experimental data for three-dimensional boundary-layer flows.

Because of the interest in the boundary-layer problem for three-dimensional flow at large as well as at small Mach numbers, it seemed desirable to present a boundary-layer momentum equation in three dimensions for a fluid having variable density and viscosity in a form analagous to the momentum equation for two-dimensional flow.

SYMBOLS

ρ	density
μ	coefficient of viscosity
t	time
p	static pressure
x,y,z	three mutually perpendicular coordinates, Cartesian system
δ	nominal thickness of boundary layer
$\bar{i},\bar{j},\bar{k}$	unit vectors along x-, y-, and z-axes, respectively
u,v,w	components in the directions of x-, y-, and z-axes, respectively, of velocity inside boundary layer
$\bar{q}$	resultant velocity vector $(\bar{i}u + \bar{j}v + \bar{k}w)$
$\bar{F}$	body force vector per unit mass $(\bar{i}F_x + \bar{j}F_y + \bar{k}F_z)$

∇ del operator $\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$

$\bar{\zeta}$ vorticity vector $\left[\bar{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \bar{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \bar{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$

u_δ, v_δ components in direction of x- and y-axes, respectively, of velocity at outer edge of boundary layer

V resultant velocity inside boundary layer

V_δ resultant velocity at edge of boundary layer

ρ_δ density at edge of boundary layer

δ^*_x boundary-layer displacement thickness composed of velocity components in x-direction $\left(\int_0^\delta \left(1 - \frac{\rho u}{\rho_\delta u_\delta}\right) dz\right)$

θ_{xx} boundary-layer momentum thickness composed of velocity components in x-direction $\left(\int_0^\delta \frac{\rho u}{\rho_\delta u_\delta}\left(1 - \frac{u}{u_\delta}\right) dz\right)$

δ^*_y boundary-layer displacement thickness composed of velocity components in y-direction $\left(\int_0^\delta \left(1 - \frac{\rho v}{\rho_\delta v_\delta}\right) dz\right)$

θ_{yy} boundary-layer momentum thickness composed of velocity components in y-direction $\left(\int_0^\delta \frac{\rho v}{\rho_\delta v_\delta}\left(1 - \frac{v}{v_\delta}\right) dz\right)$

θ_{xy} $\int_0^\delta \frac{\rho u}{\rho_\delta u_\delta}\left(1 - \frac{v}{v_\delta}\right) dz$

θ_{yx} $\int_0^\delta \frac{\rho v}{\rho_\delta v_\delta}\left(1 - \frac{u}{u_\delta}\right) dz$

α angle between direction of projection on x-y plane of resultant velocity inside boundary layer and x-axis

α_δ angle between direction of projection on x-y plane of resultant velocity at outer edge of boundary layer and x-axis

δ^* boundary-layer displacement thickness for α independent of z $\left(\int_0^\delta \left(1 - \frac{\rho V}{\rho_\delta V_\delta}\right) dz\right)$

θ boundary-layer momentum thickness for α independent of z $\left(\int_0^\delta \frac{\rho V}{\rho_\delta V_\delta}\left(1 - \frac{V}{V_\delta}\right) dz\right)$

$H = \frac{\delta^*}{\theta}$

$H_x = \frac{\delta_x^*}{\theta_{xx}}$

$H_y = \frac{\delta_y^*}{\theta_{yy}}$

τ_o resultant surface shear acting on fluid

τ_{ox} component in x-direction of surface shear acting on fluid

τ_{oy} component in y-direction of surface shear acting on fluid

s distance measured along direction of flow in cases of two-dimensional flow

r radial distance from origin, always positive $\left(\sqrt{x^2 + y^2}\right)$

x_o length of diffuser measured from fictitious intersection of streamlines

r_o constant, greater than all values of r_1, for flow in converging channel

p_{st} stagnation pressure, incompressible flow h constant length greater than δ, quantities with subscript h are equal to quantities with subscript δ

Subscripts:

o at surface of plate

1 quantities for flow in converging channel

DERIVATION

Boundary-Layer Momentum Equations for Three-Dimensional Flow

The fundamental equations of flow (reference 6) are the equation of continuity which may be written as

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot\rho\bar{q} = 0 \tag{1}$$

and the equation of motion of a fluid with variable density and viscosity which may be written as either

$$\rho \frac{D\bar{q}}{Dt} = \rho\bar{F} - \bar{\nabla}p + \mu(\bar{\nabla}\cdot\bar{\nabla})\bar{q} + \frac{\mu}{3}\bar{\nabla}(\bar{\nabla}\cdot\bar{q}) + 2(\bar{\nabla}\mu\cdot\bar{\nabla})\bar{q}$$
$$+ \bar{\nabla}\mu \times \bar{\zeta} - \frac{2}{3}(\bar{\nabla}\cdot\bar{q})\bar{\nabla}\mu \tag{2}$$

or, by using the vector identity

$$\bar{\nabla} \times \bar{\zeta} = \bar{\nabla}(\bar{\nabla}\cdot\bar{q}) - (\bar{\nabla}\cdot\bar{\nabla})\bar{q}$$

as

$$\rho \frac{D\bar{q}}{Dt} = \rho\bar{F} - \bar{\nabla}p - \mu(\bar{\nabla} \times \bar{\zeta}) + \frac{4}{3}\mu\bar{\nabla}(\bar{\nabla}\cdot\bar{q}) + 2(\bar{\nabla}\mu\cdot\bar{\nabla})\bar{q}$$
$$+ \bar{\nabla}\mu \times \bar{\zeta} - \frac{2}{3}(\bar{\nabla}\cdot\bar{q})\bar{\nabla}\mu \tag{3}$$

Because the flow is assumed to be steady

$$\frac{\partial(\;)}{\partial t} = 0$$

and hence the equation of continuity becomes

$$\bar{\nabla}\cdot\rho\bar{q} = 0 \tag{4}$$

and the acceleration vector becomes

$$\frac{D\bar{q}}{Dt} = \frac{d\bar{q}}{dt} = (\bar{q}\cdot\bar{\nabla})\bar{q}$$

In the rectangular system of coordinates shown in figure 1 and used in the subsequent analysis, the acceleration vector is:

$$\frac{d\bar{q}}{dt} = \bar{i}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \bar{j}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$+ \bar{k}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

also

$$\nabla p = \bar{i}\frac{\partial p}{\partial x} + \bar{j}\frac{\partial p}{\partial y} + \bar{k}\frac{\partial p}{\partial z}$$

$$\bar{\zeta} = \bar{\nabla}\times\bar{q} = \bar{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \bar{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \bar{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\bar{\nabla}\times\bar{\zeta} = \bar{i}\left(\frac{\partial^2 v}{\partial y\partial x} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial z\partial x}\right) + \bar{j}\left(\frac{\partial^2 w}{\partial z\partial y} - \frac{\partial^2 v}{\partial z^2} - \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x\partial y}\right)$$

$$+ \bar{k}\left(\frac{\partial^2 u}{\partial x\partial z} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y\partial z}\right)$$

$$\bar{\nabla}\cdot\bar{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{q}) = \bar{i}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z}\right) + \bar{j}\left(\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y\partial z}\right)$$

$$+ \bar{k}\left(\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 v}{\partial y\partial z} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\bar{\nabla}\mu = \bar{i}\frac{\partial\mu}{\partial x} + \bar{j}\frac{\partial\mu}{\partial y} + \bar{k}\frac{\partial\mu}{\partial z}$$

$$\bar{\nabla}\mu\cdot\nabla(\;) = \frac{\partial\mu}{\partial x}\frac{\partial(\;)}{\partial x} + \frac{\partial\mu}{\partial y}\frac{\partial(\;)}{\partial y} + \frac{\partial\mu}{\partial z}\frac{\partial(\;)}{\partial z}$$

$$(\bar{\nabla}\mu\cdot\bar{\nabla})\bar{q} = \bar{i}\left(\frac{\partial\mu}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial\mu}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial\mu}{\partial z}\frac{\partial u}{\partial z}\right) + \bar{j}\left(\frac{\partial\mu}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial\mu}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial\mu}{\partial z}\frac{\partial v}{\partial z}\right)$$

$$+ \bar{k}\left(\frac{\partial\mu}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial\mu}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial\mu}{\partial z}\frac{\partial w}{\partial z}\right)$$

$$\bar{\nabla}\mu\times\bar{\zeta} = \bar{i}\left[\frac{\partial\mu}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\partial\mu}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\right]$$

$$+ \bar{j}\left[\frac{\partial\mu}{\partial z}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \frac{\partial\mu}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

$$+ \bar{k}\left[\frac{\partial\mu}{\partial x}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - \frac{\partial\mu}{\partial y}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\right]$$

$$(\bar{\nabla}\cdot\bar{q})\bar{\nabla}\mu = \bar{i}\,\frac{\partial\mu}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\bar{j}\,\frac{\partial\mu}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$+\bar{k}\,\frac{\partial\mu}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

By use of the foregoing relations the components of the equation of motion become:

Along the x-axis:

$$\rho\left(u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}+w\,\frac{\partial u}{\partial z}\right)=\rho F_x-\frac{\partial p}{\partial x}-\mu\left(\frac{\partial^2 v}{\partial y\partial x}-\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial z\partial x}\right)$$

$$+\frac{4}{3}\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right)$$

$$+2\left(\frac{\partial\mu}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\mu}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial\mu}{\partial z}\frac{\partial u}{\partial z}\right)+\frac{\partial\mu}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$-\frac{\partial\mu}{\partial z}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-\frac{2}{3}\frac{\partial\mu}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\qquad(5)$$

Along the y-axis:

$$\rho\left(u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial y}+w\,\frac{\partial v}{\partial z}\right)=\rho F_y-\frac{\partial p}{\partial y}-\mu\left(\frac{\partial^2 w}{\partial z\partial y}-\frac{\partial^2 v}{\partial z^2}-\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}\right)$$

$$+\frac{4}{3}\mu\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right)$$

$$+2\left(\frac{\partial\mu}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial\mu}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial\mu}{\partial z}\frac{\partial v}{\partial z}\right)+\frac{\partial\mu}{\partial z}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$

$$-\frac{\partial\mu}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-\frac{2}{3}\frac{\partial\mu}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\qquad(6)$$

Along the z-axis:

$$\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho F_z-\frac{\partial p}{\partial z}-\mu\left(\frac{\partial^2 u}{\partial x\partial z}-\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial y\partial z}\right)$$
$$+\frac{4}{3}\mu\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\right)$$
$$+2\left(\frac{\partial \mu}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial \mu}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial \mu}{\partial z}\frac{\partial w}{\partial z}\right)+\frac{\partial \mu}{\partial x}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$
$$-\frac{\partial \mu}{\partial y}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)-\frac{2}{3}\frac{\partial \mu}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \tag{7}$$

In common with the boundary-layer theory for two-dimensional flow (reference 7) the quantities u, v, x, y are assumed to be of the order unity, the quantities w and z of the order δ where δ is small, and μ/ρ of the order of δ^2. It is also assumed that the radii of curvature of the plate and of the streamlines in the direction of the z-axis are large compared with the thickness of the boundary layer.

Then, when all quantities of the order of δ to the first or higher powers are neglected, the equations of motion, equations (5), (6), and (7), respectively, become:

Along the x-axis:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho F_x-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial z^2}+2\frac{\partial \mu}{\partial z}\frac{\partial u}{\partial z}-\frac{\partial \mu}{\partial z}\frac{\partial u}{\partial z} \tag{8}$$

Along the y-axis:

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho F_y-\frac{\partial p}{\partial y}+\mu\frac{\partial^2 v}{\partial z^2}+2\frac{\partial \mu}{\partial z}\frac{\partial v}{\partial z}-\frac{\partial \mu}{\partial z}\frac{\partial v}{\partial z} \tag{9}$$

Along the z-axis:

$$0=\rho F_z-\frac{\partial p}{\partial z} \tag{10}$$

If it is now assumed that the body forces are negligible compared with the pressure and viscous forces in equations (8) and (9) and that the body force in equation (10) produces a negligible static pressure gradient across the boundary layer, then equations (8), (9), and (10) become the boundary-layer equations which are:

Along the x-axis:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) \tag{11}$$

Along the y-axis:

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) \tag{12}$$

Along the z-axis:

$$0 = \frac{\partial p}{\partial z}$$

The boundary-layer momentum equation in the direction of the x-axis is obtained by integrating equation (11) in the z-direction and by using the equation of continuity, equation (4), $\overline{\nabla}\cdot\rho\overline{q} = \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$. The result is:

$$\frac{\partial}{\partial x}\int_0^{\delta} \rho u^2\, dz - u_\delta \frac{\partial}{\partial x}\int_0^{\delta} \rho u\, dz + \frac{\partial}{\partial y}\int_0^{\delta} \rho uv\, dz - u_\delta \frac{\partial}{\partial y}\int_0^{\delta} \rho v\, dz = -\delta\frac{\partial p}{\partial x} - \mu_0\left(\frac{\partial u}{\partial z}\right)_0 \tag{13}$$

The corresponding boundary-layer momentum equation in the direction of the y-axis is:

$$\frac{\partial}{\partial y}\int_0^\delta \rho v^2\, dz - v_\delta \frac{\partial}{\partial y}\int_0^\delta \rho v\, dz + \frac{\partial}{\partial x}\int_0^\delta \rho uv\, dz - v_\delta \frac{\partial}{\partial x}\int_0^\delta \rho u\, dz = -\delta\frac{\partial p}{\partial y} - \mu_o\left(\frac{\partial v}{\partial z}\right)_o \qquad (14)$$

If it is assumed that the viscous stresses are negligible outside the boundary layer, and the following notations

$$\theta_{xx} = \int_0^\delta \frac{\rho u}{\rho_\delta u_\delta}\left(1 - \frac{u}{u_\delta}\right) dz$$

$$\delta_x^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_\delta u_\delta}\right) dz$$

$$\tau_{ox} = \mu_o\left(\frac{\partial u}{\partial z}\right)_o$$

$$\theta_{yx} = \int_0^\delta \frac{\rho v}{\rho_\delta v_\delta}\left(1 - \frac{u}{u_\delta}\right) dz$$

are introduced, equation (13) may be written as:

$$\frac{\partial \rho_\delta u_\delta^2 \theta_{xx}}{\partial x} + \rho_\delta u_\delta \frac{\partial u_\delta}{\partial x} \delta_x^* + \frac{\partial \rho_\delta v_\delta u_\delta \theta_{yx}}{\partial y} + \rho_\delta v_\delta \frac{\partial u_\delta}{\partial y} \delta_y^* = \tau_{ox} \qquad (13a)$$

where the component, along the x-axis, of the equation of motion for inviscid flow with $\frac{\partial(\ )}{\partial z} = 0$

$$\rho_\delta u_\delta \frac{\partial u_\delta}{\partial x} + \rho_\delta v_\delta \frac{\partial u_\delta}{\partial y} = -\frac{\partial p}{\partial x}$$

has been used outside the boundary layer.

If it is again assumed that the viscous stresses are negligible outside the boundary layer, and the notations

$$\theta_{yy} = \int_0^\delta \frac{\rho v}{\rho_\delta v_\delta}\left(1 - \frac{v}{v_\delta}\right) dz$$

$$\delta_y^* = \int_0^\delta \left(1 - \frac{\rho v}{\rho_\delta v_\delta}\right) dz$$

$$\tau_{oy} = \mu_o \left(\frac{\partial v}{\partial z}\right)_o$$

$$\theta_{xy} = \int_0^\delta \frac{\rho u}{\rho_\delta u_\delta}\left(1 - \frac{v}{v_\delta}\right) dz$$

are introduced, equation (14) may be written as:

$$\frac{\partial \rho_\delta v_\delta^2 \theta_{yy}}{\partial y} + \rho_\delta v_\delta \frac{\partial v_\delta}{\partial y} \delta_y^* + \frac{\partial \rho_\delta u_\delta v_\delta \theta_{xy}}{\partial x} + \rho_\delta u_\delta \frac{\partial v_\delta}{\partial x} \delta_x^* = \tau_{oy} \qquad (14a)$$

where the component, along the y-axis, of the equation of motion for inviscid flow with $\frac{\partial(\ )}{\partial z} = 0$

$$\rho_\delta u_\delta \frac{\partial v_\delta}{\partial x} + \rho_\delta v_\delta \frac{\partial v_\delta}{\partial y} = -\frac{\partial p}{\partial y}$$

has been used outside the boundary layer.

If the terms are separated so that each derivative contains only one term, equations (13a) and (14a) may be written as:

$$\frac{\partial \theta_{xx}}{\partial x} + \theta_{xx}\left(\frac{H_x + 2}{u_\delta}\frac{\partial u_\delta}{\partial x} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x}\right) + \frac{v_\delta}{u_\delta}\left[\frac{\partial \theta_{yx}}{\partial y} + \theta_{yx}\left(\frac{\frac{\delta_y^*}{\theta_{yx}} + 1}{u_\delta}\frac{\partial u_\delta}{\partial y} + \frac{1}{v_\delta}\frac{\partial v_\delta}{\partial y} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial y}\right)\right] = \frac{\tau_{ox}}{\rho_\delta {u_\delta}^2} \qquad (13b)$$

$$\frac{\partial \theta_{yy}}{\partial y} + \theta_{yy}\left(\frac{H_y + 2}{v_\delta}\frac{\partial v_\delta}{\partial y} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial y}\right) + \frac{u_\delta}{v_\delta}\left[\frac{\partial \theta_{xy}}{\partial x} + \theta_{xy}\left(\frac{\frac{\delta_x^*}{\theta_{xy}} + 1}{v_\delta}\frac{\partial v_\delta}{\partial x} + \frac{1}{u_\delta}\frac{\partial u_\delta}{\partial x} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x}\right)\right] = \frac{\tau_{oy}}{\rho_\delta {v_\delta}^2} \qquad (14b)$$

Equations (13a) and (14a) or (13b) and (14b) are the three-dimensional boundary-layer momentum equations for compressible flow over a flat or slightly curved plate.

Reduction of the Equations (13) and (14) to the Equations Given by Prandtl (Reference 1)

The equations (13) and (14) can be reduced to the momentum equations given by Prandtl (reference 1) by making certain substitutions. The first substitution is to change the upper limit of the integrals from δ to h, where h is a constant length everywhere greater than δ, by using

$$\int_0^\delta (\;)\,dz = \int_0^h (\;)\,dz - \int_\delta^h (\;)\,dz$$

Equation (13) then becomes

$$\frac{\partial}{\partial x}\int_0^h \rho u^2\,dz + \frac{\partial}{\partial y}\int_0^h \rho uv\,dz - u_\delta\left(\frac{\partial}{\partial x}\int_0^h \rho u\,dz + \frac{\partial}{\partial y}\int_0^h \rho v\,dz\right) - \rho_\delta u_\delta (h - \delta)\frac{\partial u_\delta}{\partial x}$$

$$- \rho_\delta v_\delta (h - \delta)\frac{\partial u_\delta}{\partial y} + h\frac{\partial p}{\partial x} + (\delta - h)\frac{\partial p}{\partial x} = -\mu_o\left(\frac{\partial u}{\partial z}\right)_o \qquad (13c)$$

When the component, along the x-axis, of the equation of motion for inviscid flow with $\frac{\partial(\;)}{\partial z} = 0$

$$\rho_\delta u_\delta \frac{\partial u_\delta}{\partial x} + \rho_\delta v_\delta \frac{\partial u_\delta}{\partial y} = - \frac{\partial p}{\partial x}$$

is used for $z > \delta$, equation (13c) becomes

$$\frac{\partial}{\partial x} \int_0^h \rho u^2 \, dz + \frac{\partial}{\partial y} \int_0^h \rho uv \, dz - u_\delta \int_0^h \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho u}{\partial y} \right) dz + h \frac{\partial p}{\partial x} = - \mu_o \left(\frac{\partial u}{\partial z} \right)_o \tag{13d}$$

When use is made of the equation of continuity (equation (4)), equation (13d) may be written as:

$$\frac{\partial}{\partial x} \int_0^h \rho u^2 \, dz + \frac{\partial}{\partial y} \int_0^h \rho uv \, dz + u_\delta \rho_h w_h + h \frac{\partial p}{\partial x} = - \mu_o \left(\frac{\partial u}{\partial z} \right)_o \tag{13e}$$

If the density and viscosity of the fluid are now assumed to be constant, equation (13e) becomes

$$\rho \left(\frac{\partial}{\partial x} \int_0^h u^2 \, dz + \frac{\partial}{\partial y} \int_0^h uv \, dz + u_\delta w_h \right) + h \frac{\partial p}{\partial x} = - \tau_{ox} \tag{13f}$$

which is the equation (1) of reference 1.

In order to obtain equation (2) of reference 1, the assumption of constant density and viscosity is made in equation (13d) and $\frac{\partial p}{\partial x}$ is replaced by

$$\frac{\partial p}{\partial x} = -\rho_\delta u_\delta \frac{\partial u_\delta}{\partial x} - \rho_\delta v_\delta \frac{\partial v_\delta}{\partial x}$$

because Bernoulli's equation

$$p = p_{st} - \frac{\rho_\delta}{2}\left(u_\delta^2 + v_\delta^2\right)$$

is used for the flow for $z > \delta$. Equation (13d) can then be expressed in the form

$$\frac{\tau_{ox}}{\rho} = \int_0^h (u_\delta - u)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz + \int_0^h \left(u_\delta \frac{\partial u_\delta}{\partial x} - u\frac{\partial u}{\partial x}\right)dz + \int_0^h \left(v_\delta \frac{\partial v_\delta}{\partial x} - v\frac{\partial u}{\partial y}\right)dz \qquad (13g)$$

If use is now made of the fact that the applicability of Bernoulli's equation for $z > \delta$ implies that the vorticity is zero for $z > \delta$, or $\frac{\partial u_\delta}{\partial y} - \frac{\partial v_\delta}{\partial x} = 0$ for $z > \delta$, then equation (13g) becomes

$$\frac{\tau_{ox}}{\rho} = \int_0^h (u_\delta - u)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz + \int_0^h \left(u_\delta \frac{\partial u_\delta}{\partial x} - u\frac{\partial u}{\partial x}\right)dz + \int_0^h \left(v_\delta \frac{\partial u_\delta}{\partial y} - v\frac{\partial u}{\partial y}\right)dz$$

which is equation (2) of reference 1.

By a similar process equation (14) can be reduced to the momentum equation for the y-direction given by Prandtl.

Reduction of Equations (13b) and (14b) to the Two-Dimensional Momentum Equation

It is of interest to show that equations (13b) and (14b) reduce to the two-dimensional momentum equation when the flow becomes essentially two dimensional. For this reduction, the definitions for θ_{xx}, θ_{yy}, θ_{yx}, θ_{xy}, δ_x^* and δ_y^* are written as follows:

$$\theta_{xx} = \int_0^\delta \frac{\rho V \cos \alpha}{\rho_\delta V_\delta \cos \alpha_\delta}\left(1 - \frac{V \cos \alpha}{V_\delta \cos \alpha_\delta}\right) dz$$

$$\theta_{yy} = \int_0^\delta \frac{\rho V \sin \alpha}{\rho_\delta V_\delta \sin \alpha_\delta}\left(1 - \frac{V \sin \alpha}{V_\delta \sin \alpha_\delta}\right) dz$$

$$\theta_{yx} = \int_0^\delta \frac{\rho V \sin \alpha}{\rho_\delta V_\delta \sin \alpha_\delta}\left(1 - \frac{V \cos \alpha}{V_\delta \cos \alpha_\delta}\right) dz$$

$$\theta_{xy} = \int_0^\delta \frac{\rho V \cos \alpha}{\rho_\delta V_\delta \cos \alpha_\delta}\left(1 - \frac{V \sin \alpha}{V_\delta \sin \alpha_\delta}\right) dz$$

$$\delta_x^* = \int_0^\delta \left(1 - \frac{\rho V \cos \alpha}{\rho_\delta V_\delta \cos \alpha_\delta}\right) dz$$

$$\delta_y^* = \int_0^\delta \left(1 - \frac{\rho V \sin \alpha}{\rho_\delta V_\delta \sin \alpha_\delta}\right) dz$$

where (see fig. 2)

$$u = V \cos \alpha$$

$$v = V \sin \alpha$$

If α is independent of z, then $\theta_{xx} = \theta_{yy} = \theta_{yx} = \theta_{xy} = \theta$ and $\delta_x^* = \delta_y^* = \delta^*$. Equation (13b) then becomes

$$\frac{\partial\theta}{\partial x} + \theta\left(\frac{H + 2}{u_\delta}\frac{\partial u_\delta}{\partial x} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x}\right) + \frac{v_\delta}{u_\delta}\left[\frac{\partial\theta}{\partial y} + \theta\left(\frac{H + 1}{u_\delta}\frac{\partial u_\delta}{\partial y} + \frac{1}{v_\delta}\frac{\partial v_\delta}{\partial y} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial y}\right)\right] = \frac{\tau_{ox}}{\rho_\delta {u_\delta}^2}$$

or

$$\frac{\partial\theta}{\partial x} + \frac{v_\delta}{u_\delta}\frac{\partial\theta}{\partial y} + \theta\left(\frac{H + 2}{u_\delta}\frac{\partial u_\delta}{\partial x} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x} + \frac{v_\delta}{u_\delta}\frac{H + 1}{u_\delta}\frac{\partial u_\delta}{\partial y} + \frac{1}{u_\delta}\frac{\partial v_\delta}{\partial y} + \frac{v_\delta}{u_\delta}\frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial y}\right) = \frac{\tau_{ox}}{\rho_\delta {u_\delta}^2}$$

Inasmuch as $v_\delta = V_\delta \sin \alpha$, $u_\delta = V_\delta \cos \alpha$, $\tau_{ox} = \tau_o \cos \alpha$, and if α is made independent of x and y as well as of z, it follows that:

$$\frac{\partial\theta}{\partial x}\cos\alpha + \frac{\partial\theta}{\partial y}\sin\alpha + \theta\left(\frac{H+2}{V_\delta}\frac{\partial V_\delta}{\partial x}\cos\alpha + \frac{H+1}{V_\delta}\frac{\partial V_\delta}{\partial y}\sin\alpha + \frac{1}{V_\delta}\frac{\partial V_\delta}{\partial y}\sin\alpha\right.$$

$$\left. + \frac{1}{\rho_\delta}\frac{\partial\rho_\delta}{\partial x}\cos\alpha + \frac{1}{\rho_\delta}\frac{\partial\rho_\delta}{\partial y}\sin\alpha\right) = \frac{\tau_o}{\rho_\delta {V_\delta}^2}$$

Note that

$$\frac{\partial(\ )}{\partial x}\cos\alpha + \frac{\partial(\ )}{\partial y}\sin\alpha = \frac{d(\ )}{ds}$$

and therefore

$$\frac{d\theta}{ds} + \theta\left(\frac{H+2}{V_\delta}\frac{dV_\delta}{ds} + \frac{1}{\rho_\delta}\frac{d\rho_\delta}{ds}\right) = \frac{\tau_o}{\rho_\delta {V_\delta}^2} \tag{15}$$

Equation (15) is the boundary-layer momentum equation for two-dimensional compressible flow (reference 4). By the same method it can be shown that equation (14b) also reduces to equation (15) when the flow is two dimensional.

Reduction of Equations (13b) and (14b) to the Boundary-Layer Momentum Equation for Radial Flow

When the flow over a flat or slightly curved plate is such that all the velocity vectors point away from a common line perpendicular to the plate, the flow is called radial flow outward from the origin. When all the velocity vectors point toward a common line perpendicular to the plate, the flow is called radial flow inward to the origin. For such flows the momentum equations (13b) and (14b) reduce to a simple form.

In order to obtain the boundary-layer momentum equation for radial flow, the x-axis is taken along one of the radial lines and the y-axis is taken at right angles to it (fig. 3). Equation (13b) or (14b) is used together with

$$u_\delta = V_\delta \cos \alpha$$

$$v_\delta = V_\delta \sin \alpha$$

In this case $\alpha = \alpha(x,y)$ and, therefore,

$$\sin \alpha = \frac{y}{r}$$

$$\cos \alpha = \frac{x}{r}$$

$$\frac{\partial \alpha}{\partial y} = \frac{\cos \alpha}{r}$$

$$\frac{\partial \alpha}{\partial x} = -\frac{\sin \alpha}{r}$$

$$r = \sqrt{x^2 + y^2}$$

When these relations are used, the expressions for the velocity derivatives become:

$$\frac{\partial u_\delta}{\partial x} = \frac{\partial V_\delta}{\partial x} \cos \alpha + V_\delta \frac{\sin^2\alpha}{r}$$

$$\frac{\partial u_\delta}{\partial y} = \frac{\partial V_\delta}{\partial y} \cos \alpha - V_\delta \frac{\sin \alpha \cos \alpha}{r}$$

$$\frac{\partial v_\delta}{\partial x} = \frac{\partial V_\delta}{\partial x} \sin \alpha - V_\delta \frac{\sin \alpha \cos \alpha}{r}$$

$$\frac{\partial v_\delta}{\partial y} = \frac{\partial V_\delta}{\partial y} \sin \alpha + V_\delta \frac{\cos^2\alpha}{r}$$

Substituting these expressions into equations (13b) or (14b) and collecting terms results in

$H = \frac{\delta^*}{\theta}$

$$\frac{d\theta}{dr} + \theta\left(\frac{H + 2}{V_\delta}\frac{dV_\delta}{dr} + \frac{1}{\rho_\delta}\frac{d\rho_\delta}{dr} + \frac{1}{r}\right) = \frac{\tau_o}{\rho_\delta {V_\delta}^2} \tag{16}$$

with $r > 0$, where dr and V_δ are positive in the direction away from the origin. Equation (16) is the boundary-layer momentum equation for radial flow.

When the flow is radial and into the origin, it is sometimes more convenient to write the momentum equation in a form in which the velocity is considered positive when directed toward the origin and in which the radial distance increases in a positive sense toward the origin. This form may be obtained by making the substitutions

$$H = H_1$$

$$\theta = \theta_1$$

$$V_\delta = -V_{\delta_1}$$

$$\rho_\delta = \rho_{\delta_1}$$

$$\tau_o = -\tau_{o_1}$$

$$r = r_o - r_1$$

in equation (16). The result is then,

$$\frac{d\theta_1}{dr_1} + \theta_1\left(\frac{H_1 + 2}{V_{\delta_1}}\frac{dV_{\delta_1}}{dr_1} + \frac{1}{\rho_{\delta_1}}\frac{d\rho_{\delta_1}}{dr_1} + \frac{1}{r_1 - r_o}\right) = \frac{\tau_{o_1}}{\rho_\delta {V_{\delta_1}}^2} \tag{17}$$

where $r_o > r_1$.

Comparison with Kehl's Equations for Flows in Converging and Diverging Channels

It may be noted that equation (16) is also the momentum equation for the boundary layer on the part of the wall of a two-dimensional diverging channel over which the flow is radial. Equation (16) is thus applicable to the flow shown in figure 4 when the origin of the coordinate system of figure 3 is placed at the point where the radially directed streamlines of figure 4 would intersect if projected. In order to obtain the equation given by Kehl (reference 8), it is assumed that the density is constant and that the distance r is measured along the center line (fig. 4). Equation (16) then becomes

$$\frac{d\theta}{dx} + \theta\left(\frac{H + 2}{u_\delta}\frac{du_\delta}{dx} + \frac{1}{x}\right) = \frac{\tau_o}{\rho_\delta {u_\delta}^2}$$

which is the equation given by Kehl for the flow in a two-dimensional diverging channel.

Similarly, equation (17) is also the momentum equation for the boundary layer on the part of a wall of a two-dimensional converging channel over which the flow is radial. The flow shown in figure 5 may be then described by equation (17) with the use of the coordinates r_o and r_1, as shown in figure 5 . If it is assumed that the density is constant and if the distance r_1 is measured along the center line, equation (17) becomes

$$\frac{d\theta_1}{dx_1} + \theta_1\left(\frac{H_1 + 2}{u_{\delta_1}}\frac{du_{\delta_1}}{dx_1} + \frac{1}{x_1 - x_o}\right) = \frac{\tau_{o_1}}{\rho_{\delta_1} {u_{\delta_1}}^2}$$

which is the equation by Kehl for flow in a converging channel.

CONCLUSIONS

Three-dimensional boundary-layer momentum equations for a fluid with variable density and viscosity are presented in a form similar to the momentum equation for two-dimensional flow. The momentum equations can be reduced to the forms of the three-dimensional momentum equations that have been given recently by Prandtl for a fluid with constant density and viscosity. When the flow becomes two dimensional, the momentum equation first given by von Kármán results. For flow in a convergent or divergent channel the equations reduce to the equations previously given by A. Kehl for a fluid with constant density and viscosity.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., September 3, 1947

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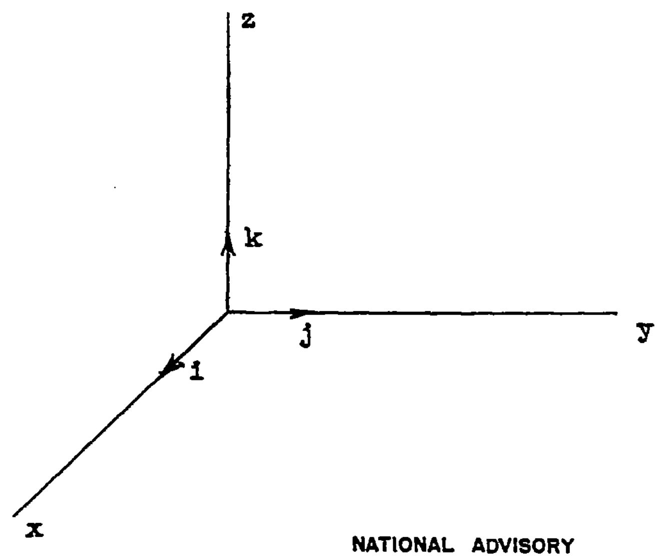

Figure 1.- Coordinate system.

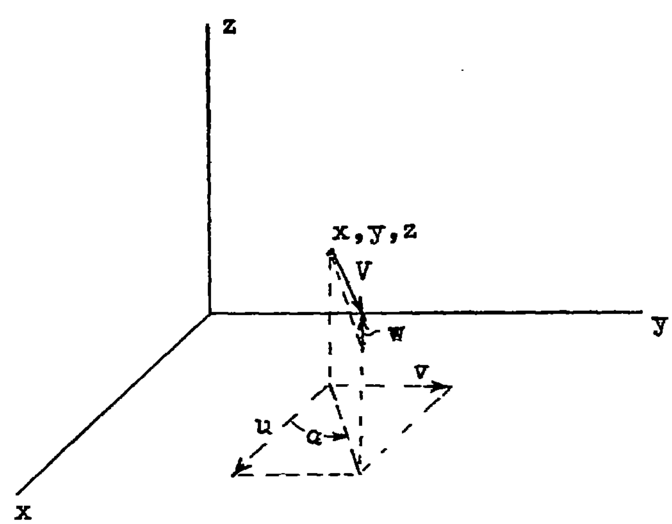

Figure 2.- Velocity components.

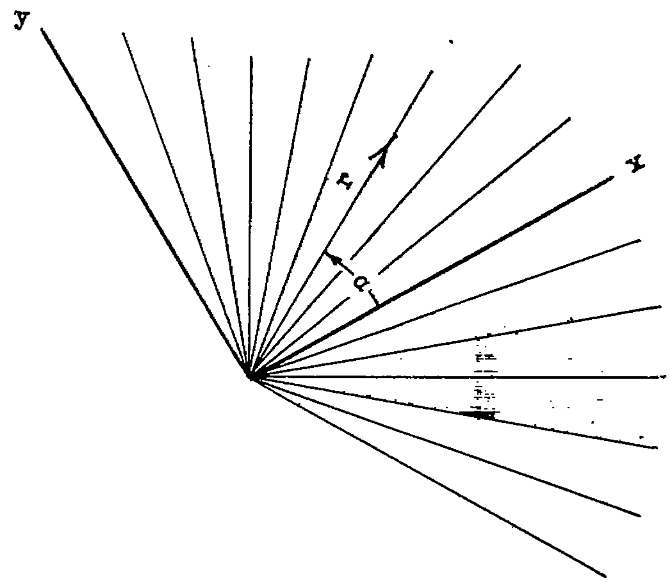

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Figure 3.- Coordinate system for radial flow.

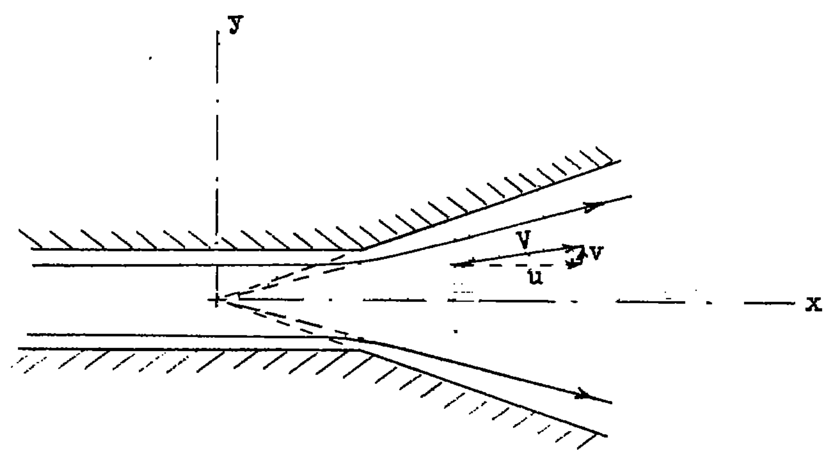

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Figure 4.- Coordinate system for flow in a diverging channel.

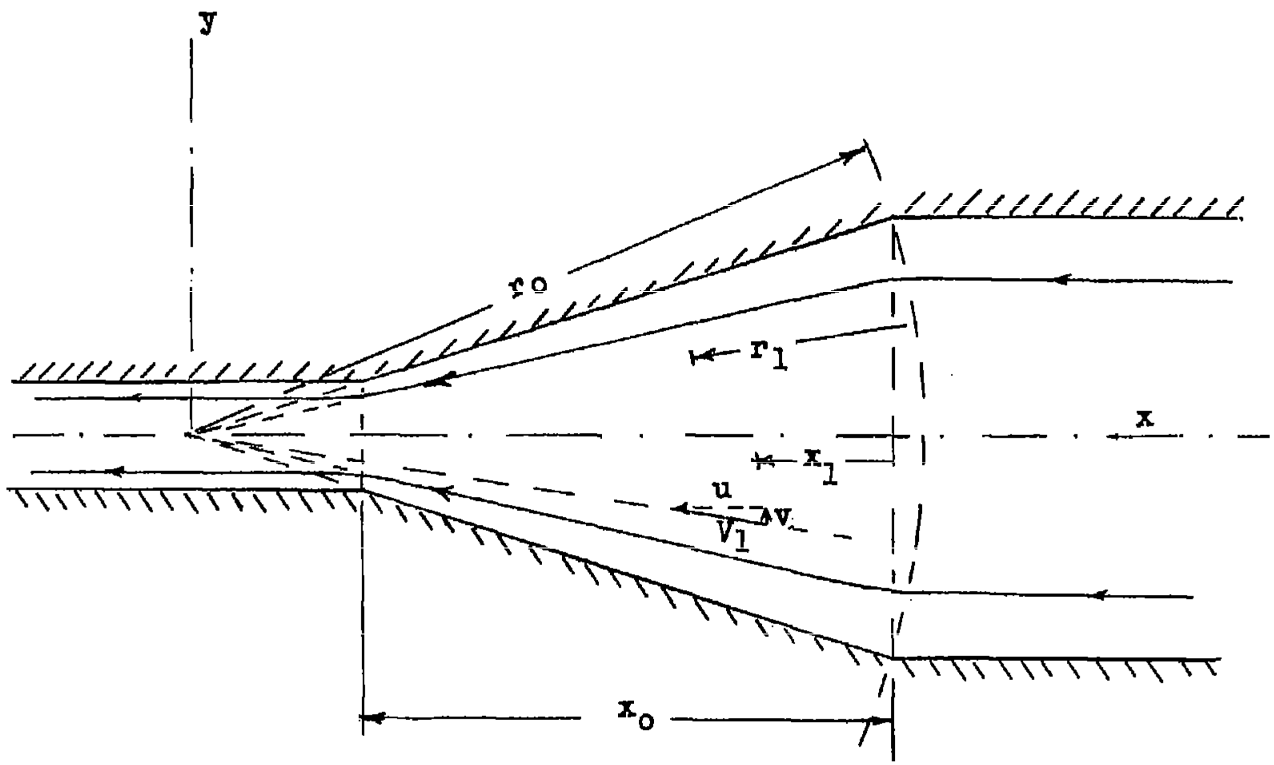

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Figure 5.- Coordinate system for flow in a converging channel.